MATH 245 F19, Exam 3 Solutions

1. Carefully define the following terms: subset, intersection, De Morgan's Law (for sets).

   Let A, B be sets. We say that A is a subset of B if every element of A is an element of B. Let A, B be sets. The intersection of A, B is the set $\{x : x \in A \wedge x \in B\}$. De Morgan's Law states: Let A, B, U be sets, with $A \subseteq U$ and $B \subseteq U$. Then $(A \cup B)^c = A^c \cap B^c$, and $(A \cap B)^c = A^c \cup B^c$.

2. Carefully define the following terms: cardinality, set of departure, irreflexive.

   Let A be a set. The cardinality of A is the number of elements that A contains. Given sets A, B, and $R \subseteq A \times B$, the set of departure of R is A. Given a relation R on a set A, we say that R is irreflexive if, for all $a \in A$, $(a, a) \notin R$.

3. Prove, using definitions, that for all sets A, B, we have $(A \cup B) \setminus (A \cap B) \subseteq A \Delta B$.

   This is half of Theorem 8.12a; citing this theorem is not a proof using definitions.

   Let $x \in (A \cup B) \setminus (A \cap B)$. Then $(x \in A \cup B) \wedge (x \notin A \cap B)$. By simplification twice, $x \in A \cup B$ and $x \notin A \cap B$. Since $x \notin A \cap B$, $\neg(x \in A \wedge x \in B)$; i.e. $x \notin A \vee x \notin B$ $(\star)$. Since $x \in A \cup B$, $x \in A \vee x \in B$. We have two cases, $x \in A$ and $x \in B$.

   Case 1: If $x \in A$, by disjunctive syllogism with $(\star)$, $x \notin B$. By conjunction, $x \in A \wedge x \notin B$, and by addition $(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)$.

   Case 2: If $x \in B$, by disjunctive syllogism with $(\star)$, $x \notin A$. By conjunction, $x \in B \wedge x \notin A$, and by addition $(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)$.

   In both cases, $(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)$ and hence $x \in A \Delta B$.

4. Prove or disprove: For all sets A, B, C with $A \subseteq B$, $B \subseteq C$, and $C \subseteq A$, we must have $A = C$.

   The statement is true, and is proved in two parts, $C \subseteq A$ and $A \subseteq C$. To prove $C \subseteq A$ is easy, as it is a hypothesis. To prove $A \subseteq C$, let $x \in A$ be arbitrary. Since $A \subseteq B$, we have $x \in B$. Since $B \subseteq C$, we have $x \in C$.

5. Prove or disprove: For all sets A, B, we must have $2^A \cup 2^B = 2^{A \cup B}$.

   The statement is false. Many counterexamples are possible, such as: Let $A = \{x\}, B = \{y, z\}$. We must prove that $2^A \cup 2^B \neq 2^{A \cup B}$, which must be done with a counterexample within this counterexample.

   METHOD 1: Set $\alpha = \{x, z\}$. $\alpha \subseteq A \cup B$, so $\alpha \in 2^{A \cup B}$. However, $\alpha \not\subseteq A$, so $\alpha \notin 2^A$. Also, $\alpha \not\subseteq B$, so $\alpha \notin 2^B$. By conjunction, $\alpha \notin 2^A \wedge \alpha \notin 2^B$. By De Morgan's Law (for propositions), $\neg(\alpha \in 2^A \vee \alpha \in 2^B)$, or $\neg(\alpha \in 2^A \cup 2^B)$. Hence $\alpha \notin 2^A \cup 2^B$.

   METHOD 2: We explicitly calculate $2^A = \{\emptyset, \{x\}\}, 2^B = \{\emptyset, \{y\}, \{z\}, \{y, z\}\}$, and $2^{A \cup B} = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}$. We also calculate $2^A \cup 2^B = \{\emptyset, \{x\}, \{y\}, \{z\}, \{y, z\}\}$. Now, we observe element $\{x, y\}$ is in $2^{A \cup B}$ but not in $2^A \cup 2^B$.

6. Prove or disprove: For all sets A, B, C with $A \subseteq B$ and $B \subseteq C$, we must have $A \times B \subseteq B \times C$.

The statement is true. Let $x \in A \times B$ be arbitrary. Then $x = (y, z)$, with $y \in A$ and $z \in B$. Because $y \in A$ and $A \subseteq B$, in fact $y \in B$. Because $z \in B$ and $B \subseteq C$, in fact $z \in C$. Hence $x = (y, z)$ with $y \in B$ and $z \in C$, so $x \in B \times C$.

7. Prove or disprove: For all sets A, B, we must have $A \times B$ equicardinal with $(A \times B) \times A$.

The statement is false. To disprove requires a counterexample. Many are possible, such as: Let $A = \{x, y\}, B = \{z\}$. There are now two ways to continue.

METHOD 1 (explicit calculation): We have $A \times B = \{(x, z), (y, z)\}$, so $|A \times B| = 2$. However, $(A \times B) \times A = \{((x, z), x), ((y, z), y), ((x, z), y), ((y, z), x)\}$, so $|(A \times B) \times A| = 4$.

METHOD 2 (theorem): We recall the theorem in the book that $|Y \times Z| = |Y||Z|$ for all finite sets Y, Z. We have $|A \times B| = |A||B| = 2 \cdot 1 = 2$, and $|(A \times B) \times A| = |A \times B||A| = 2 \cdot 2 = 4$.

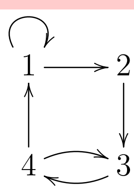
Note: It is not sufficient that our attempt to find a bijection between $A \times B$ and $(A \times B) \times A$ fails; perhaps a different attempt would succeed. For example, if $A = B = \mathbb{Z}$, then in fact the two sets *are* equicardinal.

8. Let A, B be sets with $A \subseteq B$. Prove or disprove: For all transitive relations R on B, we must have $R|_A$ also transitive.

The statement is true. Let $(x, y), (y, z) \in R|_A$. Then $x, y, z \in A$, and $(x, y), (y, z) \in R$. Since R is transitive on B, also $(x, z) \in R$. Since $x, z \in A$, in fact $(x, z) \in R|_A$.

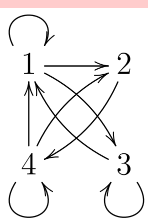
For problems 9,10: Let $R = \{(1, 1), (1, 2), (2, 3), (3, 4), (4, 1), (4, 3)\}$, a relation on $A = \{1, 2, 3, 4\}$.

9. Draw the digraph representing R. Determine, with justification, whether or not R is each of: reflexive, symmetric, and transitive.



R is not reflexive because, e.g., $(2, 2) \notin R$.
R is not symmetric because, e.g., $(2, 3) \in R$ and $(3, 2) \notin R$.
R is not transitive because, e.g., $(1, 2), (2, 3) \in R$ and $(1, 3) \notin R$.

10. Compute $R \circ R$. Give your answer both as a digraph and as a set.



$R \circ R = \{(1, 1), (1, 2), (1, 3), (2, 4), (3, 1), (3, 3), (4, 1), (4, 2), (4, 4)\}$.
Note: every missing or extra piece in a solution, will cost points.